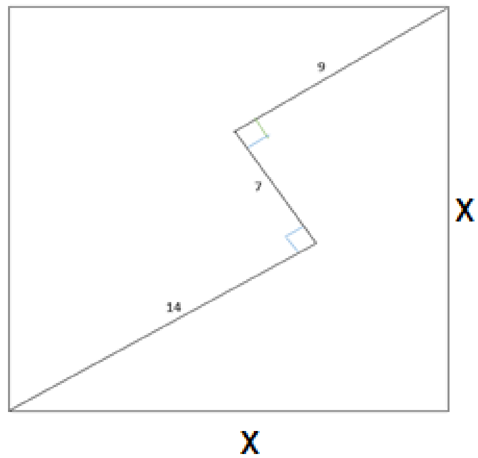
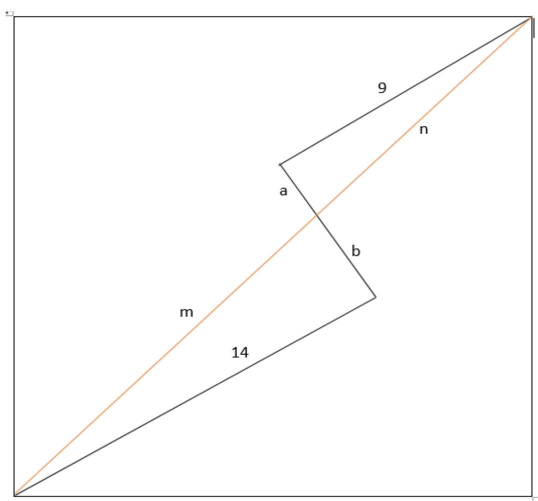
1. Find the side length "X" of the square in the following case:



Solution:

Draw the diagonal and name the segments as shown:



The two right triangles obtained are similar, therefore:

$$\frac{a}{b} = \frac{9}{14} \qquad (1)$$

Given: $a + b = 7 \qquad (2)$

Solve this system of equation (1, 2) by substituting: b = $\frac{14}{9}a$ in (2)

$a(1 + \frac{14}{9}) = \frac{23}{9}a = 7$ so: $a = \frac{63}{23}$ and: $b = \frac{98}{23}$

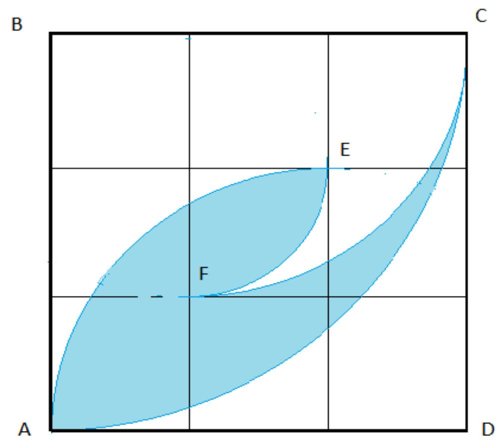
In the same triangles also we know the length of diagonal is: m + n = $\sqrt{2}$ X, in which m and n can be calculated using Pythagorean Theorem, separately:

$$m^2 = 14^2 + (\frac{98}{23})^2 = \frac{14^2.\ 23^2 + 14^2.\ 7^2}{23^2}, \quad \text{So: } m = \frac{14}{23}\sqrt{23^2 + 7^2} = \frac{14}{23}\sqrt{578} = \frac{14}{23}.17\sqrt{2}$$

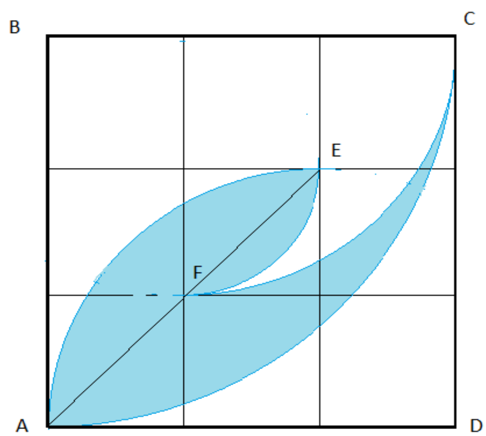
$$n^2 = 9^2 + (\frac{63}{23})^2 = \frac{9^2.\ 23^2 + 9^2.\ 7^2}{23^2}, \quad \text{So: } n = \frac{9}{23}\sqrt{23^2 + 7^2} = \frac{9}{23}\sqrt{578} = \frac{9}{23}.17\sqrt{2}$$

Therefore: $m + n = \left(\frac{14}{23} + \frac{9}{23}\right).17\sqrt{2} = 17\sqrt{2} = \sqrt{2}\,X$ Then: $X = 17$ ✓

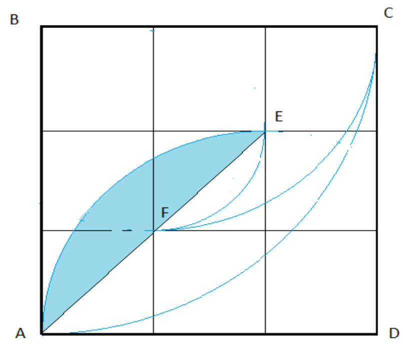
2. Calculate the shaded area in the 3x3 diagram bellow.



Solution: The line AE divides the shaded area in 3 separate areas:

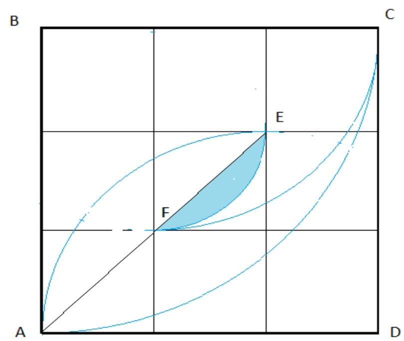


a. A quarter of circle with a radius of 2 units minus the triangle in its sector:



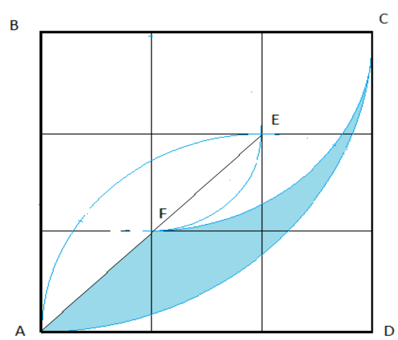
A = $\frac{\pi(2^2)}{4}$ - $\frac{1}{2}(2 \times 2)$ = $\pi - 2$

b. A quarter of circle with a radius of 1 unit minus the triangle in its sector:



B = $\frac{\pi(1^2)}{4}$ - $\frac{1}{2}(1 \times 1)$ = $\frac{\pi}{4}$ - $\frac{1}{2}$

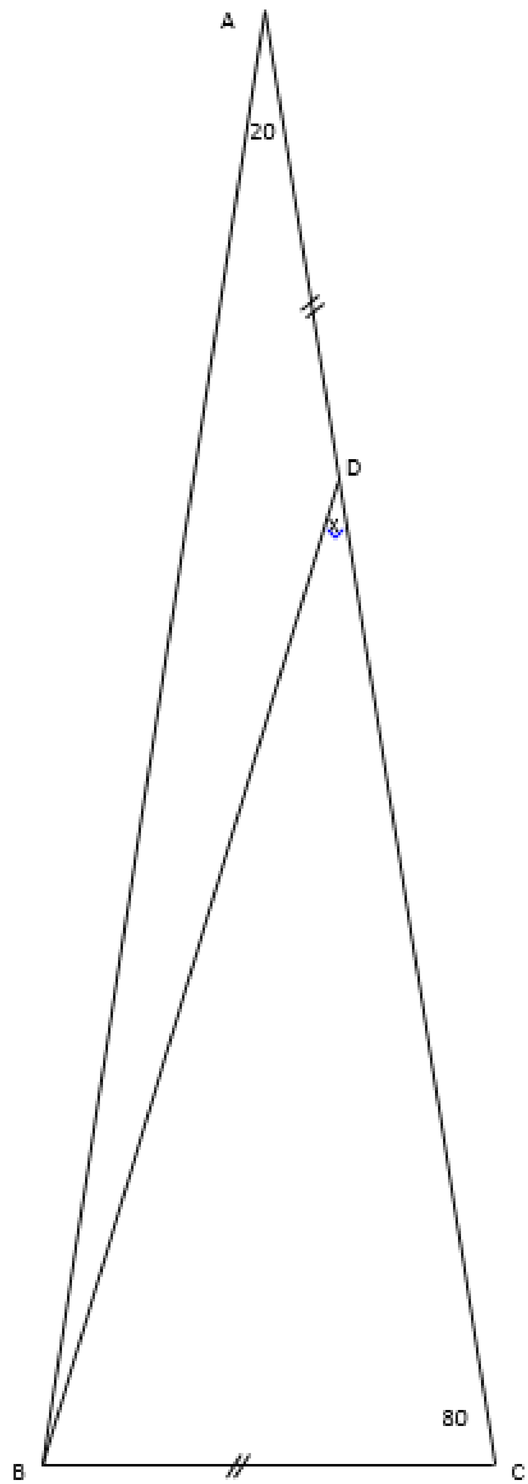
c. The area enclosed by 2 quarter of circles with R = 3 and 2, and the line AF:



C = $\frac{\pi(3^2)}{4}$ - $\frac{\pi(2^2)}{4} - \frac{3+2}{2} \times 1 = \frac{9\pi}{4}$ - $\pi - \frac{5}{2} = \frac{5\pi}{4}$ - $\frac{5}{2}$

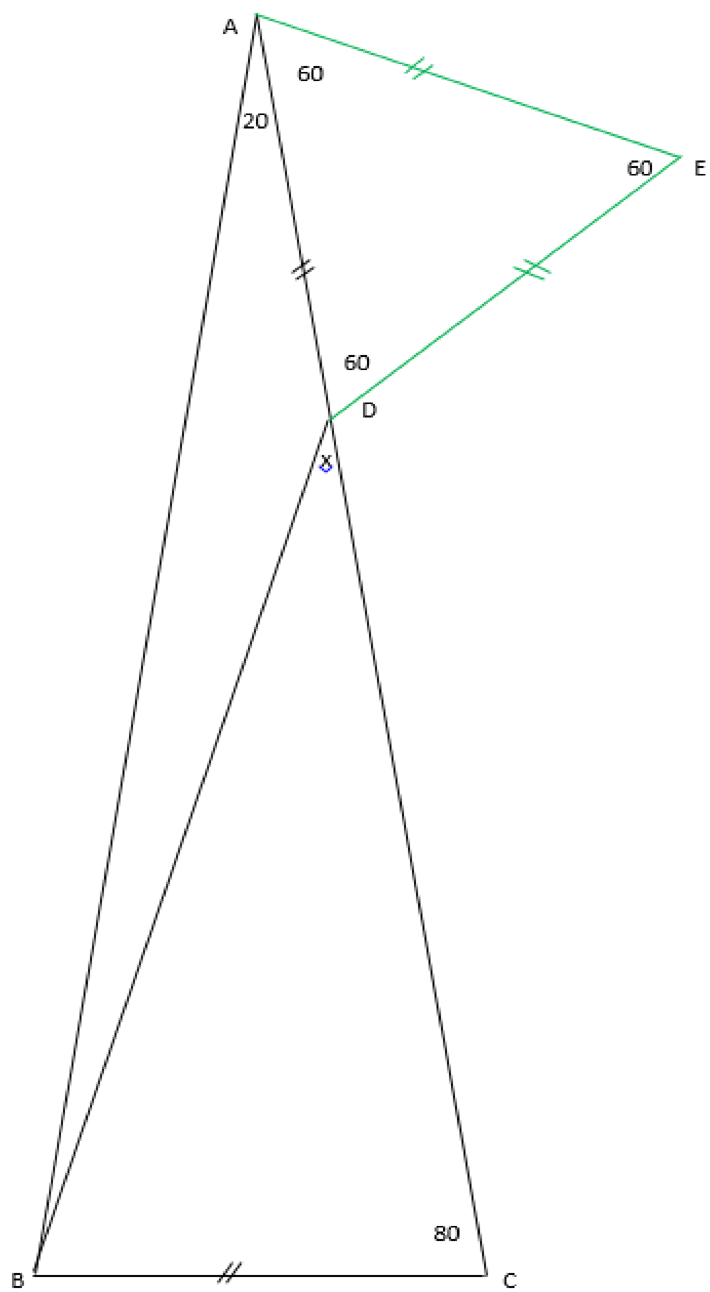
So the total shaded area is: A + B + C = $\frac{5\pi-10}{2}$ = $\frac{5}{2}$ (π - 2) ✓

3. In the triangle ABC, if A = 20˚, C = 80˚, and AD = BC. Then find the angle x?

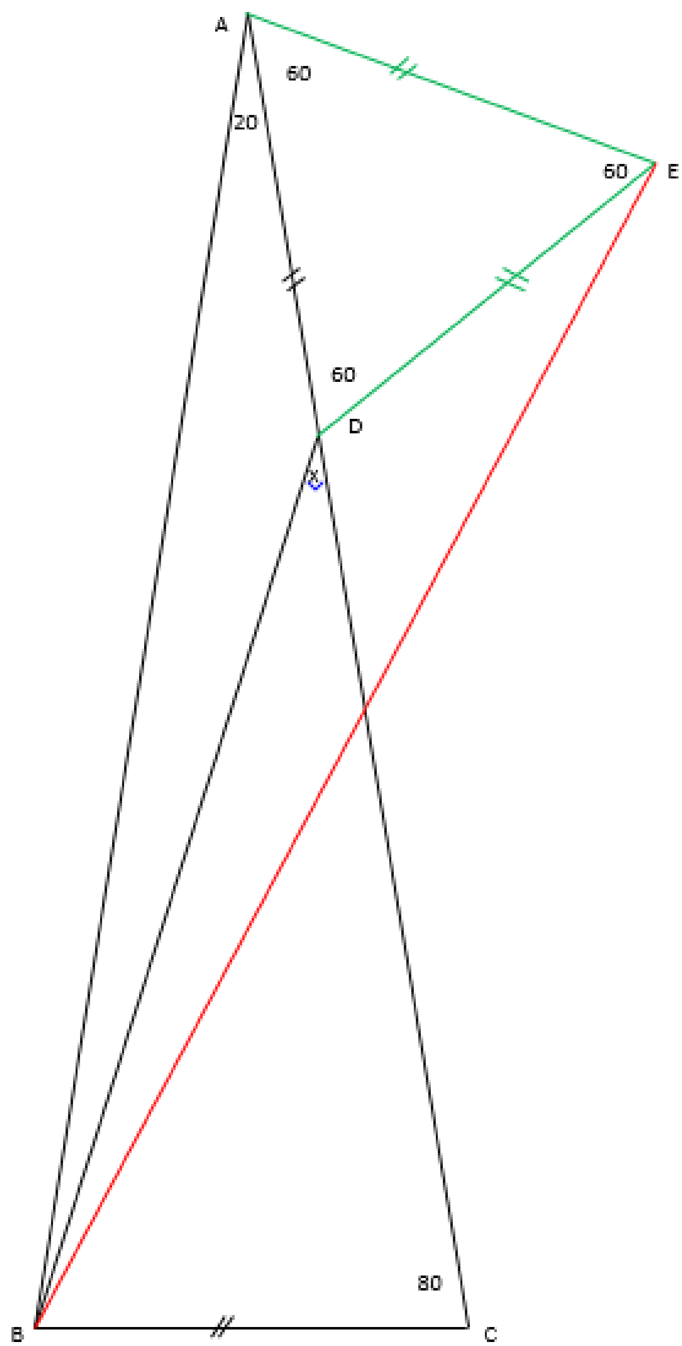


Solution: Triangle ABC is Isosceles, because the angle ABC is also 80˚ then AB = AC

Draw the equilateral tringle AED



Connect E to B. Triangles ABC and ABE are congruent, SAS, sharing side AB, AE = BC, and the angle: ABC= BAE = 80˚

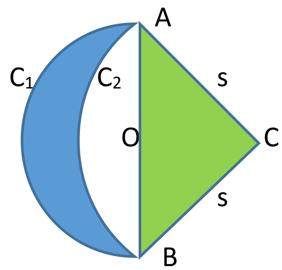


Therefore angle AEB is also 80˚. The triangle AEB is Isosceles and AB = EB so the angle DEB = 20˚.

Conclusion: triangles ABD and EDB are congruent, SSS. Therefore angles ABD = EBD =10˚ then the angle CBD is 70˚. Therefore the angle x in the triangle CBD is 180 – 80 – 70 = 30˚ ✓

4. In the following diagram, ABC is right, isosceles triangle with sides S. C_1 is a semi-circle with diameter AB centred at O midpoint of AB and C_2 is a quarter of circle with radius S and centre C.

Prove the area of the triangle is equal to the blue area enclosed by two arcs.



Solution:

The area of the triangle ABC is: $\frac{S^2}{2}$ ✓

The hypotenuse of the triangle AB = $S\sqrt{2}$, then the radius of C_1 is: $\frac{S\sqrt{2}}{2}$.

The area of the semi-circle C_1 is: $\frac{1}{2}\pi\left(\frac{S\sqrt{2}}{2}\right)^2 = \frac{\pi S^2}{4}$

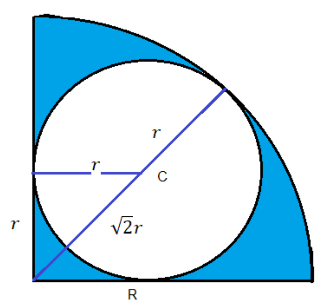
The area of the quarter circle $ACBC_2$ is: $\frac{\pi S^2}{4}$, so the area of white spot is: $\frac{\pi S^2}{4} - \frac{S^2}{2}$

Therefore the area of the blue zone is the semi-circle area minus the white part:

$$\frac{\pi S^2}{4} - \left(\frac{\pi S^2}{4} - \frac{S^2}{2}\right) = \frac{S^2}{2}$$ ✓

Which is equal to the area of the triangle.

5. In the following diagram, a circle with radius r is inscribed in a quarter of a larger circle with radius R. Find the ratio of the shaded to the inner circle area.



Solution:

$$R = r + \sqrt{2}r = r(1 + \sqrt{2}) \text{ Shown diagonally}$$

The total area of the quarter sector is: A = $\frac{\pi}{4}R^2 = \frac{\pi}{4}r^2(1+\sqrt{2})^2 = \frac{\pi}{4}r^2\left(3+2\sqrt{2}\right) = (3\frac{\pi}{4}+\frac{\pi}{2}\sqrt{2})r^2$

The area of the inner circle is: B: πr^2

So the shaded area is A – B = $(3\frac{\pi}{4}+\frac{\pi}{2}\sqrt{2})r^2 - \pi r^2 = (-\frac{\pi}{4}+\frac{\pi}{2}\sqrt{2})r^2 = \frac{\pi}{4}(2\sqrt{2}-1)r^2$

Therefore the ratio is: $\frac{2\sqrt{2}-1}{4}$ ✓

6. Find the smallest positive three-digit integer N, which has a remainder of 2 when divided by 6, a remainder of 5 when divided by 9 and a remainder of 7 when divided by 11.

Solution:

According to the divisions rule, results can be represented as:

$$\frac{N}{6} = a + \frac{2}{6}, \qquad \frac{N}{9} = b + \frac{5}{9}, \qquad \frac{N}{11} = c + \frac{7}{11}$$

In which a, b and c are also integers and respectively divisible by 6, 9 and 11 with:

The LCM of the denominator will have factors of: 2, 3, 3, and 11. Therefore the LCM is: 2.3.3.11 = 198

Refer to the division rules again, it is known if we subtract the remainder from the number N or we add the difference between the divisor and the remainder to N, the result of division will be an integer.

Note: $\frac{D\ (dividend)}{d\ (\ divisor)}$ – Q (quotient) = $\frac{R\ (remainder)}{d}$

The difference of the divisor and remainder in each case is 4, using division property that means:

$$\frac{N}{6} = \frac{198-4}{6} = a, \qquad remainder = 2$$

$$\frac{N}{9} = \frac{198-4}{9} = b, \qquad remainder = 5$$

$$\frac{N}{11} = \frac{198-4}{11} = c, \qquad remainder = 7$$

Therefore: N = 194 with: a = 34 R=2, b = 21 R=5 and c = 17 R=7 ✓

7. a, b, c are 3 consecutive terms of an arithmetic sequence, Prove:
   a. (a + b), (a + c) and (b + c) are also consecutive terms of an arithmetic sequence.
   b. $\frac{1}{\sqrt{b}+\sqrt{c}}$, $\frac{1}{\sqrt{a}+\sqrt{c}}$ and $\frac{1}{\sqrt{a}+\sqrt{b}}$ are also consecutive terms of an arithmetic sequence.

Solution:

a. Let's "a" be the first term, then "b = a+d" and "c = a+2d" will be the next terms.

Therefore: a+b = 2a+d, a+c = 2a+2d and b+c = 2a+3d. These terms make a new arithmetic sequence with the same common difference d. ✓

b. $$\frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a+d}+\sqrt{a+2d}}.\frac{\sqrt{a+d}-\sqrt{a+2d}}{\sqrt{a+d}-\sqrt{a+2d}} = \frac{\sqrt{a+d}-\sqrt{a+2d}}{a+d-a-2d} = \frac{\sqrt{a+d}-\sqrt{a+2d}}{-d} = \frac{\sqrt{a+2d}-\sqrt{a+d}}{d} \quad (1)$$

$$\frac{1}{\sqrt{a}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{a+2d}}.\frac{\sqrt{a}-\sqrt{a+2d}}{\sqrt{a}-\sqrt{a+2d}} = \frac{\sqrt{a}-\sqrt{a+2d}}{a-a-2d} = \frac{\sqrt{a}-\sqrt{a+2d}}{-2d} = \frac{\sqrt{a+2d}-\sqrt{a}}{2d} \quad (2)$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{a+d}}.\frac{\sqrt{a}-\sqrt{a+d}}{\sqrt{a}-\sqrt{a+d}} = \frac{\sqrt{a}-\sqrt{a+d}}{a-a-d} = \frac{\sqrt{a}-\sqrt{a+d}}{-d} = \frac{\sqrt{a+d}-\sqrt{a}}{d} \quad (3)$$

To prove these 3 terms represent an arithmetic sequence, calculate (3 -2) and (2 – 1):

$$\frac{\sqrt{a+d}-\sqrt{a}}{d} - \frac{\sqrt{a+2d}-\sqrt{a}}{2d} = \frac{2\sqrt{a+d}-2\sqrt{a}-\sqrt{a+2d}+\sqrt{a}}{2d} = \frac{2\sqrt{a+d}-\sqrt{a+2d}-\sqrt{a}}{2d} = D$$

$$\frac{\sqrt{a+2d}-\sqrt{a}}{2d} - \frac{\sqrt{a+2d}-\sqrt{a+d}}{d} = \frac{\sqrt{a+2d}-\sqrt{a}-2\sqrt{a+2d}+2\sqrt{a+d}}{2d} = \frac{2\sqrt{a+d}-\sqrt{a+2d}-\sqrt{a}}{2d} = D$$

D is the common difference. ✓

8. Three men—A, B, and C—crossed paths walking through woods on a cold night. They decided to light a fire to rest by, and set out to gather some firewood. A came back with 5 logs of wood, B brought 3 logs, but C came back empty-handed. C requested that they let him rest by the fire and promised to pay them some money in the morning. In the morning C paid them $8. How should A and B split the money fairly?

   a. A $7; B $1
   b. A $6; B $2
   c. A $5; B $3

d. A \$4; B \$4
e. None of these

Solution:

All three men are equally benefited by the fire from the 8 logs of wood. Each man used 8/3 logs of wood through the night. Therefore,

A contributed 5 - 8/3 = 7/3 logs of wood.

B contributed 3 - 8/3 = 1/3 log of wood, and C zero.

Therefore they must share \$8 in proportion to 7/3:1/3 and 0/3 so A \$7 and B \$1. ✓

9. Multiple identities in one question:
   If A, B and C are the 3 angles of a triangle, prove: sin2A + sin 2B +sin2C = 4 sinA . sin B . sin C (1)

Solution:

Recall to the following identities: Sin 2A = 2sinA.cosA

sin(A+B) = sinA.cosB + cosA.sinB , and cos(A-B) = cosA.cosB + sinA.sinB

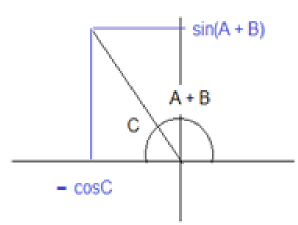
a. We prove: sin2A + sin 2B = 2 sin(A+B).cos(A-B)

RHS = 2 (sinA.cosB + cosA.sinB)(cosA.cosB + sinA.sinB)

$= 2(\sin A.\cos A.\cos^2 B + \sin B.\cos B.\sin^2 B + \sin B.\cos B.\cos^2 A + \sin A.\cos A.\sin^2 B)$

$= 2\sin A.\cos A(\cos^2 B + \sin^2 B) + 2\sin B.\cos B(\sin^2 B + .\cos^2 A) = \sin 2A + \sin 2B$ ✓

b. Given: A + B + C = π, we can see:



i. sin(A + B) = sin C ✓
ii. cos(A + B) = - cos C ✓

c. cos(A-B) - cos(A + B) = cosA.cosB + sinA.sinB – (cosA.cosB - sinA.sinB) = 2sinA.sinB ✓

d. Prove of identity (1):

According to a. LHS: sin2A + sin 2B +sin2C = 2 sin(A+B).cos(A-B) +sin2C

= 2 sin(A+B).cos(A-B) + 2sinC.cosC

= 2 sinC.cos(A-B) + 2sinC.cosC (according to b.i)

= 2 sinC.(cos(A-B) + cosC) = 2 sinC.(cos(A-B) - cos(A + B)) (according to b.ii)

Then according to c. = 2 sinC.(2sinA.sinB) = 4 sinA . sin B . sin C

Therefore: sin2A + sin 2B +sin2C = 4 sinA . sin B . sin C ✓

10. in the following sequence: $\left(1-\frac{1}{2}\right),\left(2-\frac{2}{3}\right),\left(3-\frac{3}{4}\right),\ldots\left(15-\frac{15}{16}\right)$

a. write the general term in simplest form

b. find the product of all terms

Solution:

a. $U_n = n - \frac{n}{n+1}$ ✓
b. After simplifying each term:

$$P = \left(\frac{1}{2}\right).\left(\frac{4}{3}\right).\left(\frac{9}{4}\right).\left(\frac{16}{5}\right).\left(\frac{25}{6}\right).\left(\frac{36}{7}\right)\ldots\ldots\ldots\ldots\left(\frac{225}{16}\right)$$

As we can see the numerator is the product of consecutive integers square (n^2) from 1 to 15 and the denominator is the product of consecutive integers (n+1) from 2 to 16.

After crossing out common factors from numerator with denominator we get:

$$P = 1.2.3.4.5\ldots\ldots\ldots\frac{15}{16} = \frac{15!}{16}$$ ✓